Nonparametric Bayesian System Reliability with Sets of Priors

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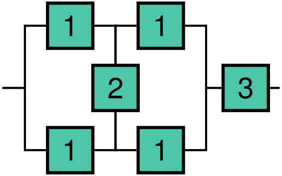
MMR Durham 2016-04-13

System Reliability



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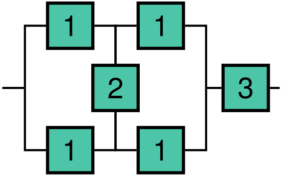

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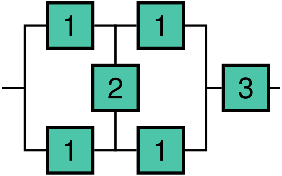

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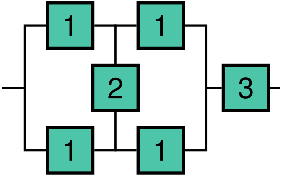

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How to combine these two information sources?

Bayesian Inference

expert info + data $\rightarrow$ complete picture

Bayesian Inference

expert info	+	data	→	complete picture
prior distribution	+	sample distribution	→	posterior distribution
$f(p)$	$\times$	$f(s \mid p)$	$\propto$	$f(p \mid s)$

► Bayes' Rule

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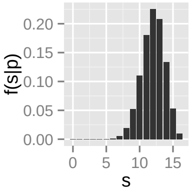
Binomial distribution

$$s \mid p \sim \text{Binomial}(n, p)$$



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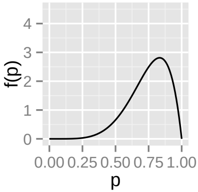

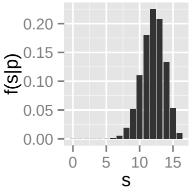

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Beta prior	Binomial distribution	Beta posterior

► conjugacy

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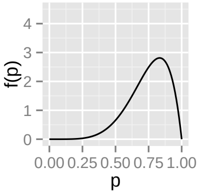

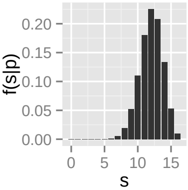

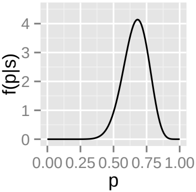

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- conjugate prior makes learning about parameter tractable, just update hyperparameters: $\alpha^{(0)} \rightarrow \alpha^{(n)}$, $\beta^{(0)} \rightarrow \beta^{(n)}$
- closed form for some inferences: $\text{E}[p \mid s] = \frac{\alpha^{(n)}}{\alpha^{(n)}+\beta^{(n)}}$

Nonparametric Component Reliability

Functioning probability p_t^k of k for each time $t \in \mathcal{T} = \{\dot{t}_1, \dot{t}_2, \ldots\}$

- discrete component reliability function $R^k(t) = p_t^k,\ t \in \mathcal{T}$.

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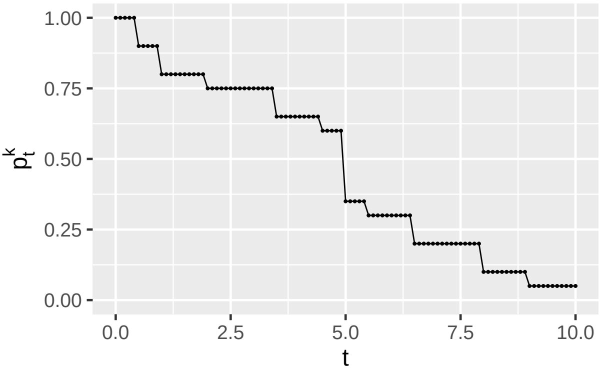

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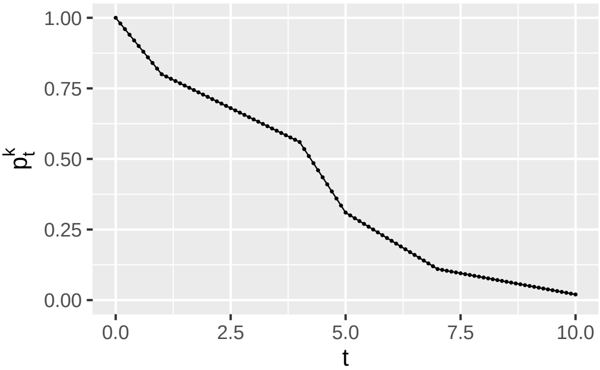

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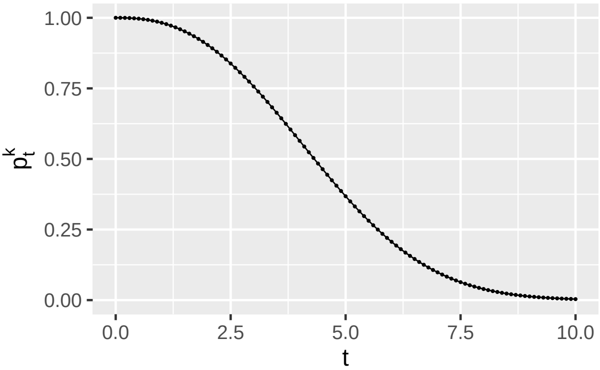

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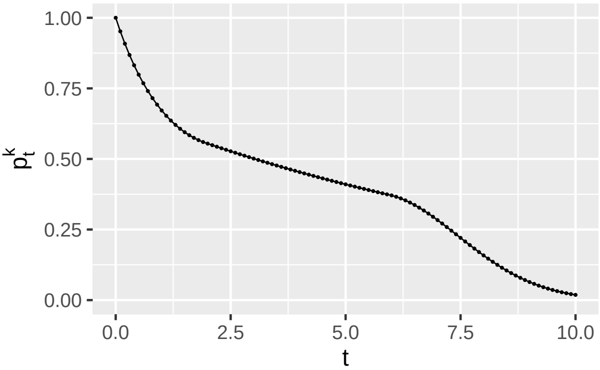

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Prior-Data Conflict

What if expert information and data tell different stories?

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- *informative prior beliefs* and *trusted data* (sampling model correct, no outliers, etc.) are in conflict
- "[...] the prior [places] its mass primarily on distributions in the sampling model for which the observed data is surprising" (Evans and Moshonov 2006)
- there are not enough data to overrule the prior

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$$n^{(n)} = n^{(0)} + n, \qquad y^{(n)} = \frac{n^{(0)}}{n^{(0)} + n} y^{(0)} + \frac{n}{n^{(0)} + n} \cdot \frac{s}{n}$$

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$\mathrm{Var}[p \mid s] = \dfrac{y^{(n)}(1 - y^{(n)})}{n^{(n)} + 1}$ decreases with n!

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Uncertainty about probability statements

smaller sets = more precise probability statements

Lottery A
Number of winning tickets:
exactly known as 5 out of 100
► $P(\text{win}) = 5/100$

Lottery B
Number of winning tickets:
not exactly known, supposedly
between 1 and 7 out of 100
► $P(\text{win}) = [1/100, 7/100]$

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- Bounds for inferences (point estimate, prediction, . . .) by min/max over

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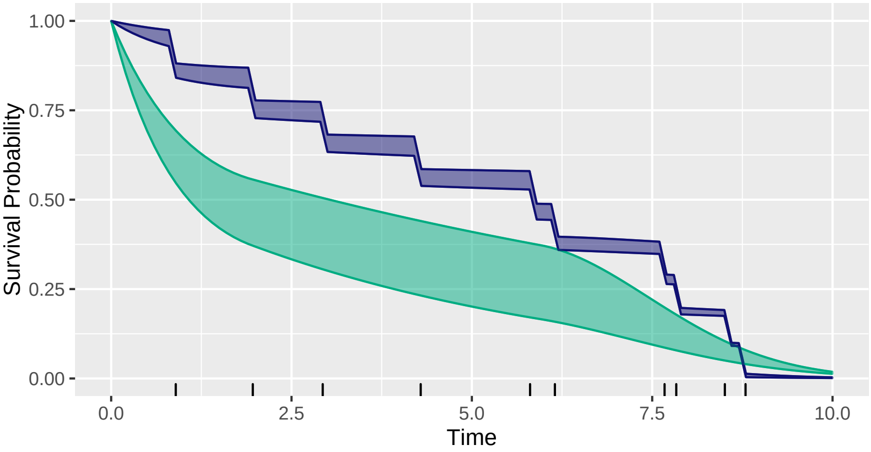

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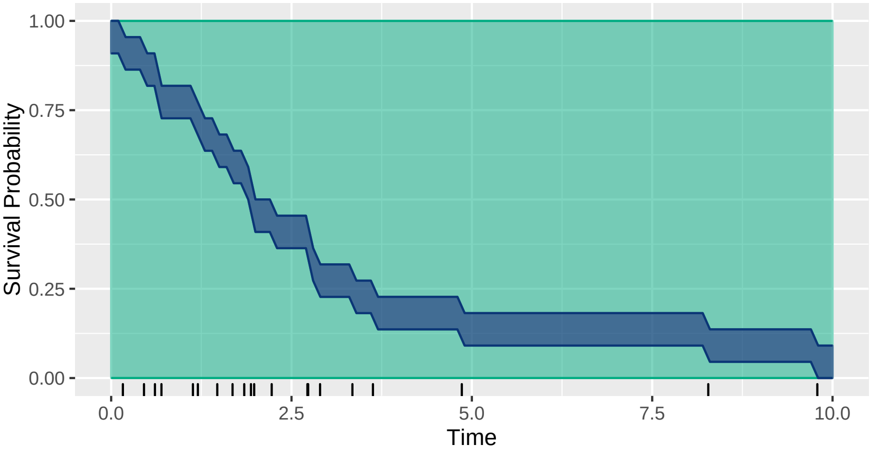

$$[\underline{n}^{(0)}, \overline{n}^{(0)}] = [1, 2]$$
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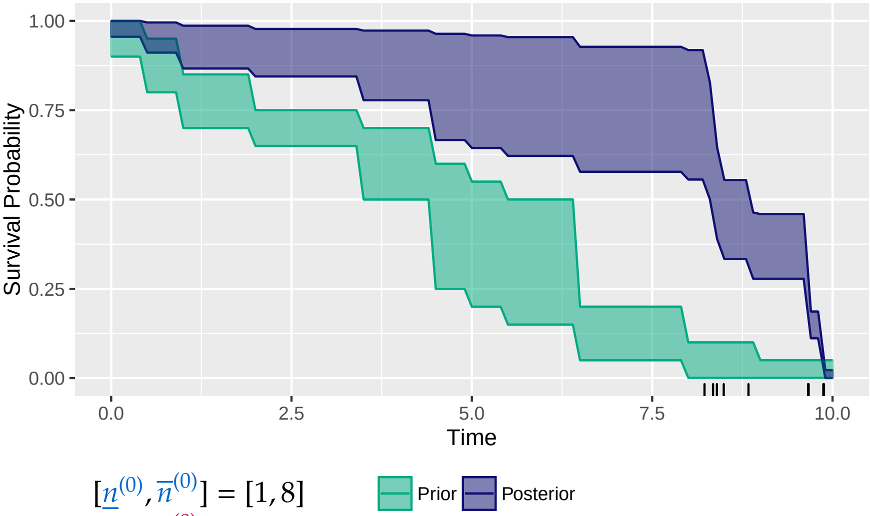

$[\underline{n}^{(0)}, \overline{n}^{(0)}] = [1, 8]$

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System Reliability

- Closed form for the system reliability via the survival signature:

$$R_{\text{sys}}\left(t \mid \bigcup_{k=1}^{K}\left\{n_{k,t}^{(0)}, y_{k,t}^{(0)}, \boldsymbol{t}^k\right\}\right) = P(T_{\text{sys}} > t \mid \cdots)$$

$$= \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \ldots, l_K) \prod_{k=1}^{K} P(C_t^k = l_k \mid n_{k,t}^{(0)}, y_{k,t}^{(0)}, \boldsymbol{t}^k)$$

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Survival signature $\Phi(l_1, \ldots, l_K)$
(Coolen and Coolen-Maturi 2012)
$= P(\text{system functions} \mid \{l_k$ k 's function$\}^{1:K})$

l_1	l_2	l_3	Φ
0	0	1	0
1	0	1	0
2	0	1	1/3
3	0	1	1
4	0	1	1

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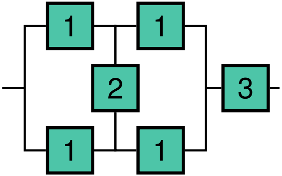

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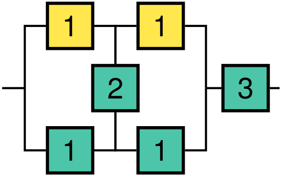

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3	0	1	1
4	0	1	1

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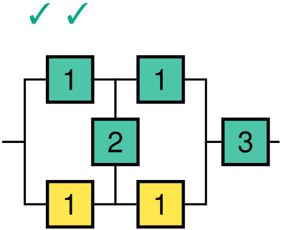

System Reliability

- Closed form for the system reliability via the survival signature:

$$R_{\text{sys}}\Big(t \mid \textstyle\bigcup_{k=1}^{K}\big\{n_{k,t}^{(0)}, y_{k,t}^{(0)}, \boldsymbol{t}^k\big\}\Big) = P(T_{\text{sys}} > t \mid \cdots)$$

$$= \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \ldots, l_K) \prod_{k=1}^{K} P(C_t^k = l_k \mid n_{k,t}^{(0)}, y_{k,t}^{(0)}, \boldsymbol{t}^k)$$

Survival signature $\Phi(l_1, \ldots, l_K)$
(Coolen and Coolen-Maturi 2012)
$= P(\text{system functions} \mid \{l_k$ k 's function$\}^{1:K})$

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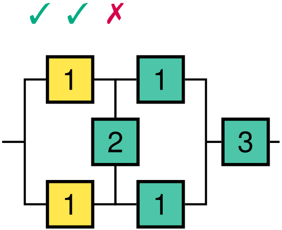

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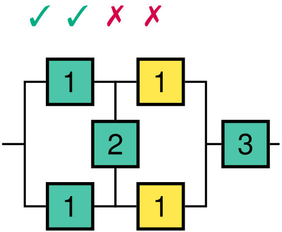

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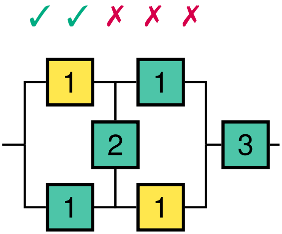

System Reliability

- Closed form for the system reliability via the survival signature:

$$
\begin{aligned}
R_{\text{sys}}\Big(t \mid \textstyle\bigcup_{k=1}^{K}\big\{n^{(0)}_{k,t}, y^{(0)}_{k,t}, \boldsymbol{t}^{k}\big\}\Big) &= P(T_{\text{sys}} > t \mid \cdots) \\
&= \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1,\ldots,l_K) \prod_{k=1}^{K} P(C^k_t = l_k \mid n^{(0)}_{k,t}, y^{(0)}_{k,t}, \boldsymbol{t}^{k})
\end{aligned}
$$

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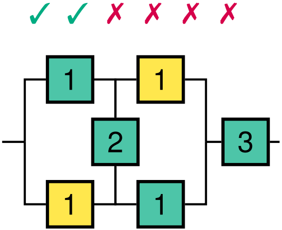

System Reliability

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Posterior predictive probability that in a new system, l_k of the m_k k's function at time t:

$$\binom{m_k}{l_k} \int [P(T < t \mid p_t^k)]^{l_k} [P(T \geq t \mid p_t^k)]^{m_k - l_k} f(p_t^k \mid n_{k,t}^{(0)}, y_{k,t}^{(0)}, \boldsymbol{t}^k)\, dp_t^k$$

- analytical solution for integral: $C_t^k \mid n_{k,t}^{(0)}, y_{k,t}^{(0)}, \boldsymbol{t}^k \sim \text{Beta-binomial}$

System Reliability Bounds

- Bounds for $R_{\text{sys}}\Big(t \,\Big|\, \bigcup_{k=1}^{K} \big\{n_{k,t}^{(0)}, y_{k,t}^{(0)}, \boldsymbol{t}^k\big\}\Big)$ over $\bigcup_{k=1}^{K} \Big\{ \qquad \Big\}$:
  - $\min R_{\text{sys}}(\cdot)$ by $y_{k,t}^{(0)} = \underline{y}_{k,t}^{(0)}$ for any $n_{k,t}^{(0)}$ (Walter, Aslett, and Coolen 2016, Theorem 1)

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System Reliability Bounds

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  - numeric optimization over $[\underline{n}_{k,t}^{(0)}, \overline{n}_{k,t}^{(0)}]$ in the very few cases where Theorem 2 & Lemma 3 do not apply

System Reliability Bounds

- Bounds for $R_{\text{sys}}\Big(t \,\Big|\, \bigcup_{k=1}^{K} \big\{n_{k,t}^{(0)}, y_{k,t}^{(0)}, \boldsymbol{t}^k\big\}\Big)$ over $\bigcup_{k=1}^{K} \big\{ \qquad \big\}$:
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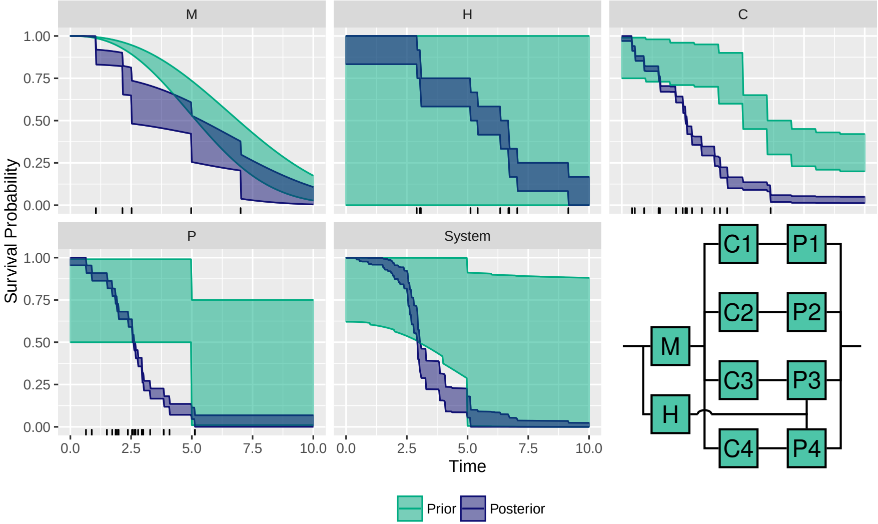
  - numeric optimization over $[\underline{n}_{k,t}^{(0)}, \overline{n}_{k,t}^{(0)}]$ in the very few cases where Theorem 2 & Lemma 3 do not apply
  - implemented in **R** package `ReliabilityTheory` (Aslett 2016)

System Reliability Bounds



Summary & Outlook

Summary:

- Nonparametric modeling of component reliability curves
- Bayesian model combining expert knowledge and test data
- Set of system reliability functions reflects uncertainties from limited data, vague expert information, and prior-data conflict
- Easy-to-use implementation in **R** package `ReliabilityTheory` (Aslett 2016)

Summary & Outlook

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- Set of system reliability functions reflects uncertainties from limited data, vague expert information, and prior-data conflict
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Next steps:

- Allow right-censored observations (RUL estimation)
- Allow dependence between components (common-cause failure, ...)
- Use for system design (where to put extra redundancy?)
- Use for maintenance planning

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